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LETTER TO THE EDITOR

Multifractal wavefunction at the localisation threshold

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Abstract. We point out that the field theoretical description of the localisation transition exhibits anomalous scaling for the moments of the probability density at the localisation threshold. This property hints at a multifractal structure of the wavefunction. We attempt an analysis of the corresponding singularity distribution.

One of the most exciting features of strange objects appearing in condensed matter physics is multifractality (Mandelbrot 1974, Frisch and Parisi 1983, Benzi *et al* 1984), namely the coexistence of several populations of singularities in measure, each occupying a set of non-trivial fractal dimension. This feature has been observed in such diverse objects as the energy dissipating set in fully developed turbulence (Frisch and Parisi 1983, Benzi *et al* 1984), strange attractors in dynamical systems (Benzi *et al* 1984, 1985, Halsey *et al* 1986, Jensen *et al* 1985, Paladin and Vulpiani 1986) and the growth site distribution in diffusion-limited aggregates (Turkevitch and Scher 1985, Halsey *et al* 1985). A similar feature is shared by the anomalous voltage distribution in random resistor networks at the percolation threshold (de Arcangelis *et al* 1985). One usually makes this property apparent by considering the anomalous scaling behaviour of the moments of the measure describing the mass distribution of the object, coarse grained over cells of linear size l, in the limit of small l. Following Jensen *et al* (1985), let us consider a set of N points of the object, sampled with a probability proportional to its mass density. We can estimate the probability $p_i(l)$ that a point chosen at random belongs to the ith cell of linear size l, by computing N_i/N, where N_i is the number of points belonging to the ith cell, in the limit of large N. We then compute the qth moment of $p_i(l)$ by averaging over the cells:

$$\Gamma(q, l) = \langle p_i(l)^{q-1} \rangle \sim l^{\tau(q)}. \tag{1}$$

In the case of a homogeneous fractal of fractal dimension D one has

$$\tau(q) = (q-1)D. \tag{2}$$

Deviations of $\tau(q)$ from the linear behaviour (2) indicate the appearance of multifractality. If one assumes that the fractal can be described as an interwoven family of singularities of type α (where α is defined by $p_i(\alpha) \sim l^{\alpha}$ around a singularity of type α), each distributed over a set of fractal dimension $f(\alpha)$, it is possible to relate $f(\alpha)$ to $\tau(q)$ by means of a Legendre transformation (Frisch and Parisi 1983, Benzi *et al*

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1984, Hentschel and Procaccia 1983, Halsey *et al* 1986). The interest of this transformation lies in the universality of $f(\alpha)$ and of the range $[\alpha_{\min}, \alpha_{\max}]$ to which α may belong.

We wish to point out in this letter that one finds anomalous scaling of the moments of the probability distribution in the context of the theory of Anderson localisation. Wegner (1980) showed indeed, by means of field theoretical renormalisation group techniques, that the quantities $P^{(q)}(E)$ (defined below), which can be considered as some form of these moments, scale with an infinite hierarchy of exponents which are not simply related to each other. The $P^{(q)}(E)$, considered as a function of the energy E near the localisation threshold E_c, are defined by

$$P^{(q)}(E) = \overline{\sum_{\lambda} |\psi_\lambda(\boldsymbol{r})|^{2q}\delta(E - e_\lambda)}/\rho(E) \tag{3}$$

where $\psi_\lambda(\boldsymbol{r})$ is the amplitude of the localised wavefunction $|\lambda\rangle$ with energy e_λ at site $\boldsymbol{r}$ in a tight-binding model with a random potential, $\rho(E)$ is the density of states per site and energy, and the average is taken with respect to all realisations of the potential. He obtained the following behaviour of the $P^{(q)}$:

$$P^{(q)}(E) \sim (\xi(E))^{-x_q} \tag{4}$$

where ξ (which itself behaves like $(E_c - E)^{-\nu}$) is the localisation length. The exponents x_q are given by

$$x_q = (q-1)d - q(q-1)\varepsilon + \mathrm{O}(\varepsilon^2). \tag{5}$$

Here d denotes the dimensionality of ambient space, which is taken to be equal to $2+\varepsilon$, with a small ε. One obtains therefore an infinite sequence of independent exponents, contrary to the usual situation in critical phenomena. This is related to the existence of an infinite number of relevant perturbations to the fixed point Hamiltonian of the non-linear σ model in two dimensions (Brézin *et al* 1976a).

In the following we attempt to relate the behaviour of the $P^{(q)}(E)$ to a multifractal structure of the wavefunction near the localisation threshold. Let us make the hypothesis that only one 'typical' wavefunction φ_E contributes essentially to $P^{(q)}(E)$. Since φ_E is different from zero for length scales up to ξ and then rapidly decays, we can relate $P^{(q)}(E)$ to the averaged moments of the wavefunction, summed over a box of linear size ξ:

$$P^{(q)}(E) \sim \xi^{-x_q} \sim \sum_{\boldsymbol{r}\in \mathrm{box}(\xi)} |\varphi_E(\boldsymbol{r})|^{2q}. \tag{6}$$

In the spirit of scaling theory, the same quantity, evaluated for a probability density coarse grained over a box of linear size l, should scale like $(\xi/l)^{-x_q}$, since the lattice constant is l times larger. This allows us to identify x_q (equation (5)) with $\tau(q)$ (equation (1)). If we now extend (5) to encompass non-integer values of q, we apply a Legendre transformation to obtain $f(\alpha)$:

$$\begin{aligned} \alpha &= \mathrm{d}\tau(q)/\mathrm{d}q \\ f(\alpha) &= \tau(q) - q\,\mathrm{d}\tau/\mathrm{d}q. \end{aligned} \tag{7}$$

We thus obtain the result

$$f(\alpha) = d - (4\varepsilon)^{-1}(d+\varepsilon-\alpha)^2 + \mathrm{O}(\varepsilon^2). \tag{8}$$

This function has the characteristic parabolic shape of that found by Jensen *et al* (1985), and its maximum value is d, the space dimensionality, as it was reasonable to expect, since the support of the function should be the whole lattice (up to distances of order ξ). One should however reject values of $f(\alpha)$ smaller than zero, given its interpretation as a fractal dimension. One would then retrieve information on the allowed range of α, were it not for the fact that the bounds of this range correspond to values of q too large (positive and negative) for the result of the ε expansion (5) to be trusted.

Let us remark that the same non-linear σ model with an n-component order parameter which, in the limit $n \to 0$, describes Anderson localisation describes instead, for $n > 2$, a Heisenberg-type critical point. In this case, the quantities analogous to the $P^{(q)}(E)$ are (up to a normalisation factor) the moments of the σ (i.e. the z component of the order parameter). In this case, however, anomalous scaling is not observed. The scaling quantities correspond to the operators (Brézin *et al* 1976b)

$$O_m = C_m^{(n/2)-1}(\sigma) \tag{9}$$

where $C_m^{(n/2)-1}(x)$ are Gegenbauer polynomials orthogonal for the measure $(1-x^2)^{(n-3)/2}$ on the interval $(-1, 1)$. These scaling quantities are characterised by the scaling exponents

$$\zeta_m = -\frac{m(m+n-2)}{2}\frac{\varepsilon}{n-2} + \mathrm{O}(\varepsilon^2). \tag{10}$$

The moments of σ are linear combinations of these scaling quantities. Their behaviour is dominated by the largest exponent. When $n > 2$ and m is an integer, the largest ζ_m corresponds to the smallest value of m, which is zero for a generic polynomial even in σ, or one for a polynomial odd in σ. On the other hand, when $n < 2$ (and therefore for $n \to 0$, which is the case of localisation), the largest ζ_m corresponds to the largest value of m, and $P^{(q)}(E)$ corresponds to a polynomial of degree $2q$. All different exponents are therefore expected to appear.

This letter suggests that it would be worthwhile to reconsider the numerical evidence proposed by Soukoulis and Economou (1984) (and challenged by Siebesma and Pietronero (1985)) for a fractal structure of the wavefunction in the localisation problem, in the light of the concepts of multifractality. Another interesting piece of numerical work which has some bearing on the present subject is due to Ioffe *et al* (1985), who have shown some evidence for an anomalous scaling behaviour of the moments of the participation ratio near the localisation threshold. It is difficult to relate these quantities to the $P^{(q)}(E)$, but this is a direction worth pursuing.

We have shown, in conclusion, that some known results of the field theoretical approach to Anderson localisation hint at a multifractal nature of the wavefunction at the localisation threshold. This opens the way for the first time to an understanding of multifractality on the basis of analytical renormalisation group methods.

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